

ON SEQUENCES WITH PRESCRIBED METRIC DISCREPANCY BEHAVIOR

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ABSTRACT. An important result of H. Weyl states that for every sequence $(a_n)_{n \geq 1}$ of distinct positive integers the sequence of fractional parts of $(a_n \alpha)_{n \geq 1}$ is uniformly distributed modulo one for almost all α . However, in general it is a very hard problem to calculate the precise order of convergence of the discrepancy D_N of $(\{a_n \alpha\})_{n \geq 1}$ for almost all α . By a result of R. C. Baker this discrepancy always satisfies $ND_N = \mathcal{O}(N^{\frac{1}{2} + \varepsilon})$ for almost all α and all $\varepsilon > 0$. In the present note for arbitrary $\gamma \in (0, \frac{1}{2}]$ we construct a sequence $(a_n)_{n \geq 1}$ such that for almost all α we have $ND_N = \mathcal{O}(N^\gamma)$ and $ND_N = \Omega(N^{\gamma - \varepsilon})$ for all $\varepsilon > 0$, thereby proving that any prescribed metric discrepancy behavior within the admissible range can actually be realized.

1. INTRODUCTION

H. Weyl [12] proved that for every sequence $(a_n)_{n \geq 1}$ of distinct positive integers the sequence $(\{a_n \alpha\})_{n \geq 1}$ is uniformly distributed modulo one for almost all reals α . Here, and in the sequel, $\{\cdot\}$ denotes the fractional part function. The speed of convergence towards the uniform distribution is measured in terms of the discrepancy, which – for an arbitrary sequence $(x_n)_{n \geq 1}$ of points in $[0, 1)$ – is defined by

$$D_N = D_N(x_1, \dots, x_N) = \sup_{0 \leq a < b \leq 1} \left| \frac{\mathcal{A}_N([a, b))}{N} - (b - a) \right|,$$

where $\mathcal{A}_N([a, b)) := \#\{1 \leq n \leq N \mid x_n \in [a, b)\}$. For a given sequence $(a_n)_{n \geq 1}$ it is usually a very hard and challenging problem to give sharp estimates for the discrepancy D_N of $(\{a_n \alpha\})_{n \geq 1}$ valid for almost all α . For general background on uniform distribution theory and discrepancy theory see for example the monographs [6, 9].

A famous result of R. C. Baker [3] states that for any sequence $(a_n)_{n \geq 1}$ of distinct positive integers for the discrepancy D_N of $(\{a_n \alpha\})_{n \geq 1}$ we have

$$(1) \quad ND_N = \mathcal{O}\left(N^{\frac{1}{2}} (\log N)^{\frac{3}{2} + \varepsilon}\right) \quad \text{as } N \rightarrow \infty$$

for almost all α and for all $\varepsilon > 0$.

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Note that (1) is a general upper bound which holds for *all* sequences $(a_n)_{n \geq 1}$; however, for some specific sequences the precise typical order of decay of the discrepancy of $(\{a_n \alpha\})_{n \geq 1}$ can differ significantly from the upper bound in (1). The fact that (1) is essentially optimal (apart from logarithmic factors) as a general result covering all possible sequences can for example be seen by considering so-called lacunary sequences $(a_n)_{n \geq 1}$, i.e., sequences for which $\frac{a_{n+1}}{a_n} \geq 1 + \delta$ for a fixed $\delta > 0$ and all n large enough. In this case for D_N we have

$$\frac{1}{4\sqrt{2}} \leq \limsup_{N \rightarrow \infty} \frac{ND_N}{\sqrt{2N \log \log N}} \leq c_\delta$$

for almost all α (see [10]), which shows that the exponent $1/2$ of N on the right-hand side of (1) cannot be reduced for this type of sequence. For more information concerning possible improvements of the logarithmic factor in (1), see [5].

Quite recently in [2] it was shown that also for a large class of sequences with polynomial growth behavior Baker's result is essentially best possible. For example, the following result was shown there: Let $f \in \mathbb{Z}[x]$ be a polynomial of degree larger or equal to 2. Then for the discrepancy D_N of $(\{f(n)\alpha\})_{n \geq 1}$ for almost all α and for all $\varepsilon > 0$ we have

$$ND_N = \Omega\left(N^{\frac{1}{2}-\varepsilon}\right).$$

On the other hand there is the classical example of the Kronecker sequence, i.e., $a_n = n$, which shows that the actual metric discrepancy behavior of $(\{a_n \alpha\})_{n \geq 1}$ can differ vastly from the general upper bound in (1). Namely, for the discrepancy of the sequence $(\{n\alpha\})_{n \geq 1}$ for almost all α and for all $\varepsilon > 0$ we have

$$(2) \quad ND_N = \mathcal{O}\left(\log N (\log \log N)^{1+\varepsilon}\right),$$

which follows from classical results of Khintchine in the metric theory of continued fractions (for even more precise results, see [11]). The estimate (2) of course also holds for $a_n = f(n)$ with $f \in \mathbb{Z}[x]$ of degree 1. In [2] further examples for $(a_n)_{n \geq 1}$ were given, where $(a_n)_{n \geq 1}$ has polynomial growth behavior of arbitrary degree, such that for the discrepancy of $(\{a_n \alpha\})_{n \geq 1}$ we have

$$ND_N = \mathcal{O}\left((\log N)^{2+\varepsilon}\right)$$

for almost all α and for all $\varepsilon > 0$; see there for more details.

These results may seduce to the hypothesis that for all choices of $(a_n)_{n \geq 1}$ for the discrepancy of $(\{a_n \alpha\})_{n \geq 1}$ for almost all α we either have

$$(3) \quad ND_N = \mathcal{O}(N^\varepsilon)$$

or

$$(4) \quad ND_N = \Omega\left(N^{\frac{1}{2}-\varepsilon}\right).$$

This hypothesis, however, is wrong as was shown in [1]: Let $(a_n)_{n \geq 1}$ be the sequence of those positive integers with an even sum of digits in base 2, sorted in increasing order; that

is $(a_n)_{n \geq 1} = (3, 5, 6, 9, 10, \dots)$. Then for the discrepancy of $(\{a_n \alpha\})_{n \geq 1}$ for almost all α we have

$$ND_N = \mathcal{O}(N^{\kappa+\varepsilon})$$

and

$$ND_N = \Omega(N^{\kappa-\varepsilon})$$

for all $\varepsilon > 0$, where κ is a constant with $\kappa \approx 0,404$. Interestingly, the precise value of κ is unknown; see [8] for the background.

The aim of the present paper is to show that the example above is not a singular counter-example, but that indeed “everything” between (3) and (4) is possible. More precisely, we will show the following theorem.

Theorem 1. *Let $0 < \gamma \leq \frac{1}{2}$. Then there exists a strictly increasing sequence $(a_n)_{n \geq 1}$ of positive integers such that for the discrepancy of the sequence $(\{a_n \alpha\})_{n \geq 1}$ for almost all α we have*

$$ND_N = \mathcal{O}(N^\gamma)$$

and

$$ND_N = \Omega(N^{\gamma-\varepsilon})$$

for all $\varepsilon > 0$.

2. PROOF OF THE THEOREM

For the proof we need an auxiliary result which easily follows from classical work of H. Behnke [4].

Lemma 1. *Let $(e_k)_{k \geq 1}$ be a strictly increasing sequence of positive integers. Let $\varepsilon > 0$. Then for almost all α there is a constant $K(\alpha, \varepsilon) > 0$ such that for all $r \in \mathbb{N}$ there exist $M_r \leq e_r$ such that for the discrepancy of the sequence $(\{n^2 \alpha\})_{n \geq 1}$ we have*

$$M_r D_{M_r} \geq K(\alpha, \varepsilon) \sqrt{\frac{e_r}{(\log e_r)^{1+\varepsilon}}}.$$

Proof. For $\alpha \in \mathbb{R}$ let $a_k(\alpha)$ denote the k -th continued fraction coefficient in the continued fraction expansion of α . Then it is well-known that for almost all α we have $a_k(\alpha) = \mathcal{O}(k^{1+\varepsilon})$ for all $\varepsilon > 0$. Let $\varepsilon > 0$ be given and let α and $c(\alpha, \varepsilon)$ be such that

$$(5) \quad a_k(\alpha) \leq c(\alpha, \varepsilon) k^{1+\varepsilon}$$

for all $k \geq 1$.

Let q_l the l -th best approximation denominator of α . Then

$$(6) \quad q_{l+1} \leq (c(\alpha, \varepsilon) l^{1+\varepsilon} + 1) q_l.$$

Since $q_l \geq 2^{\frac{l}{2}}$ in any case, we have $l \leq \frac{2 \log q_l}{\log 2}$, and we obtain

$$(7) \quad q_{l+1} \leq c_1(\alpha, \varepsilon) q_l (\log q_l)^{1+\varepsilon},$$

for an appropriate constant $c_1(\alpha, \varepsilon)$. In [4] it was shown in Satz XVII that for every real α we have

$$\left| \sum_{n=1}^N e^{2\pi i n^2 \alpha} \right| = \Omega \left(N^{\frac{1}{2}} \right).$$

Indeed, if we follow the proof of this theorem we find that even the following was shown: For every α and for every best approximation denominator q_l of α there exists an $Y_l < \sqrt{q_l}$ such that $\left| \sum_{n=1}^{Y_l} e^{2\pi i n^2 \alpha} \right| \geq c_{\text{abs}} \sqrt{q_l}$. Here c_{abs} is a positive absolute constant (not depending on α).

Let now $r \in \mathbb{N}$ be given and let l be such that $q_l \leq e_r < q_{l+1}$, and let $M_r := Y_l$ from above. Then by (6) and (7) we obtain, for an appropriate constant $c_2(\alpha, \varepsilon)$,

$$\begin{aligned} \left| \sum_{n=1}^{M_r} e^{2\pi i n^2 \alpha} \right| &\geq c_{\text{abs}} \sqrt{q_l} \\ &\geq c_2(\alpha, \varepsilon) \sqrt{\frac{q_{l+1}}{(\log q_l)^{1+\varepsilon}}} \\ &\geq c_2(\alpha, \varepsilon) \sqrt{\frac{e_l}{(\log e_l)^{1+\varepsilon}}}. \end{aligned}$$

By the fact that (see Chapter 2, Corollary 5.1 of [9])

$$M_r D_{M_r} \geq \frac{1}{4} \left| \sum_{n=1}^{M_r} e^{2\pi i n^2 \alpha} \right|,$$

which is a special case of Koksma's inequality, the result follows. \square

Now we are ready to prove the main theorem.

Proof of the Theorem. Let $(m_j)_{j \geq 1}$ and $(e_j)_{j \geq 1}$ be two strictly increasing sequences of positive integers, which will be determined later. We will consider the following strictly increasing sequence of positive integers, which will be our sequence $(a_n)_{n \geq 1}$:

$$\begin{aligned} &1, 2, 3, \dots, \underbrace{m_1}_{=:A_1}, \\ &m_1 + 1^2, m_1 + 2^2, m_1 + 3^2, m_1 + 4^2, \dots, \underbrace{m_1 + e_1^2}_{=:B_1}, \\ &B_1 + 1, B_1 + 2, B_1 + 3, \dots, \underbrace{B_1 + m_2}_{=:A_2}, \\ &A_2 + 1^2, A_2 + 2^2, A_2 + 3^2, A_2 + 4^2, \dots, \underbrace{A_2 + e_2^2}_{=:B_2}, \\ &B_2 + 1, B_2 + 2, B_2 + 3, \dots, \underbrace{B_2 + m_3}_{=:A_3}, \end{aligned}$$

$$\begin{aligned}
& A_3 + 1^2, A_3 + 2^2, A_3 + 3^2, A_3 + 4^2, \dots, \underbrace{A_3 + e_3^2}_{=:B_3}, \\
& \vdots
\end{aligned}$$

Furthermore, let

$$F_s := \sum_{i=1}^s m_i + \sum_{i=1}^{s-1} e_i \quad \text{and} \quad E_s := \sum_{i=1}^s m_i + \sum_{i=1}^s e_i.$$

The sequence $(a_n)_{n \geq 1}$ is constructed in such a way that it contains sections where it grows like $(n)_{n \geq 1}$ as well as sections where it grows like $(n^2)_{n \geq 1}$. By this construction we exploit both the strong upper bounds for the discrepancy of $(\{n\alpha\})_{n \geq 1}$ and the strong lower bounds for the discrepancy of $(\{n^2\alpha\})_{n \geq 1}$, in an appropriately balanced way, in order to obtain the desired discrepancy behavior of the sequence $(\{a_n\alpha\})_{n \geq 1}$. In our argument we will repeatedly make use of the fact that

$$(8) \quad D_N(x_1, \dots, x_N) = D_N(\{x_1 + \beta\}, \dots, \{x_N + \beta\})$$

for arbitrary $x_1, \dots, x_N \in [0, 1]$ and $\beta \in \mathbb{R}$, which allows us to transfer the discrepancy bounds for $(\{n\alpha\})_{n \geq 1}$ and $(\{n^2\alpha\})_{n \geq 1}$ directly to the shifted sequences $(\{(M+n)\alpha\})_{n \geq 1}$ and $(\{(M+n^2)\alpha\})_{n \geq 1}$ for some integer M .

Let α be such that it satisfies (5) with $\varepsilon = \frac{1}{2}$. Then it is also well-known (see for example [9]) that for the discrepancy D_N of the sequence $(\{n\alpha\})_{n \geq 1}$ we have

$$(9) \quad ND_N \leq \bar{c}_1(\alpha) (\log N)^{\frac{3}{2}}$$

for all $N \geq 2$.

By the above mentioned general result of Baker, that is by (1), we know that for almost all α for the discrepancy D_N of the sequence $(\{n^2\alpha\})_{n \geq 1}$ we have

$$ND_N \leq c_3(\alpha, \varepsilon) N^{\frac{1}{2}} (\log N)^{\frac{3}{2} + \varepsilon}$$

for all $\varepsilon > 0$ and for all $N \geq 2$, for an appropriate constant $c_3(\alpha, \varepsilon)$. Actually an even slightly sharper estimate was given for the special case of the sequence $(\{n^2\alpha\})_{n \geq 1}$ by Fiedler, Jurkat and Körner in [7], who proved that

$$(10) \quad ND_N \leq c_4(\alpha, \varepsilon) N^{\frac{1}{2}} (\log N)^{\frac{1}{4} + \varepsilon}$$

for almost all α and for all $\varepsilon > 0$ and all $N \geq 2$.

Assume that α satisfies (10) with $\varepsilon = \frac{1}{8}$. Then

$$(11) \quad ND_N \leq \bar{c}_2(\alpha) N^{\frac{1}{2}} (\log N)^{\frac{3}{8}}$$

for all $N \geq 2$. Now for such α and for arbitrary N we consider the discrepancy D_N of the sequence $(\{a_n\alpha\})_{n \geq 1}$.

Case 1.

Let $N = F_l$ for some l . Then $ND_N \leq E_{l-1}D_{E_{l-1}} + (N - E_{l-1})D_{E_{l-1}, F_l}$, where $D_{x,y}$ denotes the discrepancy of the point set $(\{a_n\alpha\})_{n=x+1, x+2, \dots, y}$. Hence by (8), (9) and by the trivial estimate $D_{B_{l-1}} \leq 1$ we have

$$\begin{aligned} ND_N &\leq E_{l-1} + \bar{c}_1(\alpha) (\log m_l)^{\frac{3}{2}} \\ &\leq 2(\log m_l)^2 \\ &\leq 2(\log N)^2 \end{aligned}$$

for all l large enough, provided that (condition (i)) m_l is chosen such that $(\log m_l)^2 \geq E_{l-1}$.

Case 2.

Let $F_l < N \leq E_l$ for some l . Then by Case 1 and by (8) and (11) we have for l large enough that

$$\begin{aligned} ND_N &\leq F_l D_{F_l} + (N - F_l) D_{F_l, N} \\ &\leq 2(\log F_l)^2 + \bar{c}_2(\alpha) (N - F_l)^{\frac{1}{2}} (\log(N - F_l))^{\frac{3}{8}}. \end{aligned}$$

Note that $0 < N - F_l < e_l$.

We choose (condition (ii))

$$(12) \quad e_l := \left\lceil \frac{F_l^{2\gamma}}{\log(F_l^{2\gamma})} \right\rceil.$$

Note that conditions (i) and (ii) do not depend on α . Now assume that l is so large that $2(\log F_l)^2 < \frac{F_l^\gamma}{2}$. Then

$$\frac{F_l^\gamma}{2} \leq 2(\log F_l)^2 + (e_l \log e_l)^{\frac{1}{2}} \leq 2F_l^\gamma$$

and (note that $\gamma \leq \frac{1}{2}$)

$$(13) \quad F_l < N \leq E_l = F_l + e_l \leq 2F_l.$$

Hence

$$\begin{aligned} ND_N &\leq \max(1, \bar{c}_2(\alpha)) 2F_l^\gamma \\ &\leq \max(1, \bar{c}_2(\alpha)) 2^{1+\gamma} N^\gamma. \end{aligned}$$

Case 3.

Let $E_l < N < F_{l+1}$ for some l . Then by Case 2 and by (8) and (9) we have

$$\begin{aligned} ND_N &\leq E_l D_{E_l} + (N - E_l) D_{E_l, N} \\ &\leq E_l^\gamma + \bar{c}_1(\alpha) (\log(N - E_l))^2 \end{aligned}$$

$$\leq 2N^\gamma$$

for N large enough.

It remains to show that for every $\varepsilon > 0$ we have $ND_N \geq N^{\gamma-\varepsilon}$ for infinitely many N . Let l be given and let $M_l \leq e_l$ with the properties given in Lemma 1. Let $N := F_l + M_l$. Then by Lemma 1, Case 1, (8), (12) and (13) for l large enough we have

$$\begin{aligned} ND_N &\geq M_l D_{F_l, N} - F_l D_{F_l} \\ &\geq K(\alpha, \varepsilon) \sqrt{\frac{e_l}{(\log e_l)^{1+\varepsilon}}} - 2(\log m_l)^2 \\ &\geq \frac{F_l^\gamma}{(\log F_l)^3} \\ &\geq N^{\gamma-\varepsilon}. \end{aligned}$$

This proves the theorem. □

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